

# The energy budget in Rayleigh-Bénard convection

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It is shown using three series of Rayleigh number simulations of varying aspect ratio  $AR$  and Prandtl number  $Pr$  that the normalized dissipation at the wall, while significantly greater than 1, approaches a constant dependent upon  $AR$  and  $Pr$ . It is also found that the peak velocity, not the mean square velocity, obeys the experimental scaling of  $Ra^{0.5}$ . The scaling of the mean square velocity is closer to  $Ra^{0.46}$ , which is shown to be consistent with experimental measurements and the numerical results for the scaling of  $Nu$  and the temperature if there are strong correlations between the velocity and temperature.

This Letter will analyze the energy budget in three-dimensional simulations of Rayleigh-Bénard convection with the objective of testing theoretical assumptions used to explain the laboratory observation of non-classical heat flux (Nusselt number  $Nu$ ) exponents in classical turbulent Rayleigh-Bénard [1,2]. The budget equation for the average kinetic energy  $q^2/2$  as a function of height  $z$  [3] is

$$\frac{1}{2} \frac{\partial}{\partial t} \overline{q^2} = -\overline{pw}_{,z} - \overline{0.5wq^2}_{,z} + \overline{Ra w \theta} + Pr \overline{\partial_z^2 q^2} - Pr \epsilon(z) \quad (1)$$

where shear production  $-\overline{uw(dU/dz)}$  is part of the turbulent production term  $-0.5wq^2_{,z}$ . Based upon the observation of strong shears in the boundary layer [4,5] and large-scale flows [6], it was first assumed that the dissipation is primarily concentrated in the boundary layer and is turbulent [7]. More recently it has been suggested that the boundary layer is laminar and the distribution of the total energy dissipation is Rayleigh number dependent [8]. Both of these approaches predict crossovers in scaling behavior. Another theory [9] makes mixing layer assumptions and does not predict these crossovers. The objective in this Letter is to look at the basis for these assumptions using numerical simulations.

Assuming that  $Nu = \overline{w\theta}/\kappa d\Theta/dz \sim Ra^{\beta_T}$ , the classical exponent is  $\beta_T = 1/3$ , with suggestions since the mid-60's [12,13] that there might be corrections to this exponent. The original experimental result [1] showing that there are significant corrections has recently been extended and refined to give  $Nu \sim Ra^{-3.09}$  over nearly 10 decades of Rayleigh number  $Ra$  [2]. However, detailed experimental information such as budgets cannot be obtained from these large  $Ra$  experiments. The only information available besides temperature statistics at a single point is that the Reynolds number based upon

vertical velocity fluctuations  $w$  midway up a sidewall goes as  $Re = wd/\nu \sim Ra^{1/2}$ . These two observations are incompatible with standard turbulent parameterizations because the total dissipation is constrained to be  $\epsilon_T = Ra(Nu - 1) \sim Ra^{1.31}$ , while the standard turbulent prediction of  $\epsilon_T = Re^3$  gives  $\epsilon_T \sim Ra^{1.5}$ .

One reason a laminar boundary layer has been suggested is that this incompatibility can be resolved if the laminar relationship for dissipation is used,  $\epsilon_T \sim Re^{5/2} \sim Ra^{1.28}$ . This would suggest that convective flows are not filled with cascading eddies, but are instead filled with strong local laminar shears, including a laminar boundary layer. One way to possibly determine whether the boundary layer is laminar or turbulent is to divide the total energy dissipation  $\epsilon_T$  at some arbitrary boundary layer thickness  $\lambda_{BL}$  into the dissipation in the boundary layers  $\epsilon_{BL}$  and the dissipation in the bulk  $\epsilon_B$ . A new theory for convective scaling [8] has assumed that in a turbulent boundary layer that  $\epsilon_{BL}$  will be the same order or greater than  $\epsilon_B$ , while if the boundary layer is laminar then  $\epsilon_{BL}/\epsilon_B$  will decrease as  $Ra$  increases. This is justified using the scaling laws above for dissipation in laminar and turbulent boundary layers.

Since the average (or total) dimensionless dissipation  $\epsilon_T = Ra(Nu - 1)$  is increasing with  $Ra$ , it is possible that even if the boundary layer is laminar at one  $Ra$ , that an  $Ra$  could be reached where the boundary layer becomes unstable and turbulent. Then for higher  $Ra$ , as in a shear driven turbulent boundary layer,  $\epsilon_{BL}/\epsilon_B \rightarrow \text{constant}$  would appear.

One way to obtain detailed information and test these properties is to use simulations. Simulations have reproduced most of the scaling laws and statistical properties of the higher Rayleigh number experiments [14,15], including  $\beta_T$  between  $2/7$  and  $1/3$ . One of the properties that simulations can determine are thermal and velocity boundary layer thicknesses. The thermal boundary layer thickness  $\lambda_T$  can be determined using either  $1/Nu$  or the position of the peak of temperature fluctuations  $\overline{\theta^2}$ . Based upon experience from analysis of classical shear driven boundary layers, three definitions of the velocity boundary layer thickness  $\lambda_{BL}$  that can be calculated for convection use the velocity to give  $z^* = 1/Re$ , use the wall shear stress  $\tau = \overline{\partial u / \partial x}$  to give  $z^+ = \sqrt{\tau}$ , and use the position of the peak of the horizontal velocity fluctuations  $\lambda_u$ . The dimensionless forms have been used in these definitions.

These and additional definitions of the boundary layer thickness could be used to define the dissipation in the boundary layer  $\epsilon_{BL}$ , which we would like to relate to the dissipation at the wall. While  $\epsilon_{BL}$  depends upon

the definition of the thickness of the boundary layer, a relationship that could hold for all the definitions above is

$$\epsilon_{BL}/\epsilon_T \leq (\epsilon_W \lambda_{BL}/(d\epsilon_T)) \quad (2)$$

where  $\epsilon_W$  is the dissipation at the wall. This relationship is not very restrictive, but if  $\epsilon_W/\epsilon_T$  were bounded, because  $\lambda_{BL}$  is a decreasing function of  $Ra$  for all the definitions given above,  $\epsilon_{BL}/\epsilon_T$  would be bounded.

In a classical shear-driven boundary layer, all of the definitions of  $\lambda_{BL}$  given above scale with  $Re$  in the same manner. A major difference in convection simulations [14,15] is that each definition of  $\lambda_{BL}$  has a separate power law dependence on  $Ra$ . That is, if

$$\lambda_u \sim Ra^{-\beta_u} \quad \text{and} \quad z^+ = \sqrt{\tau} \sim Ra^{-\beta^+} \quad (3)$$

then  $\beta_u \approx 1/7$  and  $\beta^+ > \beta_u$  has been found [14]. Because these results and the theories all indicate that  $\lambda_{BL}$  is decreasing rapidly with  $Ra$ , by applying (2) both the turbulent and the laminar boundary layer theories predict that  $\epsilon_W/\epsilon_T$  should increase as  $Ra$  to a power law.

Three simulations have been analysed to determine the dependence of  $\epsilon_W/\epsilon_T$  upon  $Ra$ . The cases to be discussed are  $AR = 4$  and  $Pr = 0.3$  for  $10^5 < Ra < 10^7$ ,  $AR = 4$  and  $Pr = 7$  for  $10^4 < Ra < 10^7$  from earlier work [15], and  $AR = 1$  and  $Pr = 0.7$  for  $10^6 < Ra < 8 \times 10^7$  where Prandtl number  $Pr = \nu/\kappa$  and aspect ratio  $AR = \text{width}/\text{height}$ . The numerics [15] are pseudospectral, using Chebyshev polynomials in the vertical to provide more resolution and no-slip, constant temperature boundary conditions at top and bottom walls. In the horizontal, sines and cosines are used to represent free-slip, insulating walls. Profiles will be shown only for the  $AR = 1$  case to save space and because this case is new.

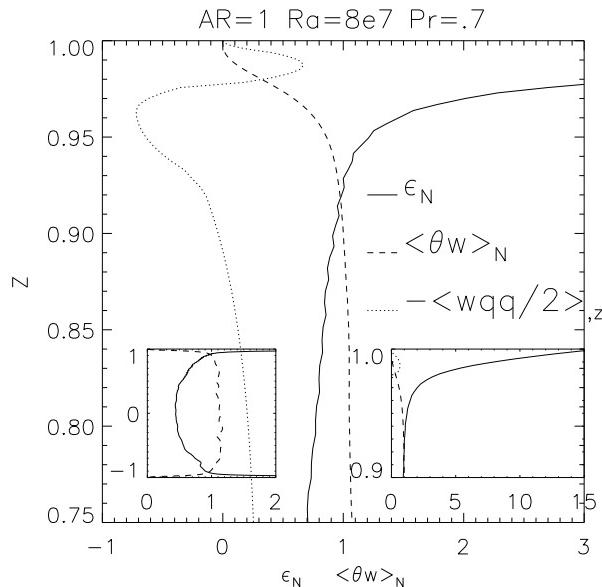


FIG. 1. Production, dissipation and turbulent transport of kinetic energy from (1) for the simulation  $AR = 1$ ,  $Pr = 0.7$  and  $Ra = 8 \times 10^7$ . All dimensionless terms are normalized by  $RaNu$  so that the production across the center  $w\theta_N$ , which is also the heat flux, is 1. Insets show production and dissipation through the entire box (height  $d = 2$ ) and very near the wall, where dissipation is very large. The difference between production and dissipation  $w\theta(z) - \epsilon(z)$  is the total transport term. As discussed, the pressure transport is close to the difference between the production and dissipation and so is much larger than the turbulent transport  $wq^2/2_z$ .

Previous work [14] demonstrated that  $\epsilon_W$  was much larger than the average dissipation and that  $\epsilon_W/\epsilon_T$  was increasing with  $Ra$ . Fig. 1 shows some of the terms in eq. (1) for  $AR = 1$ ,  $Pr = 0.7$  and  $Ra = 8 \times 10^7$ , all normalized by  $RaNu$ . For example,  $\epsilon_N(z) = \epsilon(z)/RaNu$ . The  $AR = 4$  cases are similar except that  $\epsilon_{WN} = \epsilon_W/RaNu$  is smaller,  $\min(\epsilon_N(z))$  is closer to 1, and  $wq^2/2_z$  is even smaller than here.

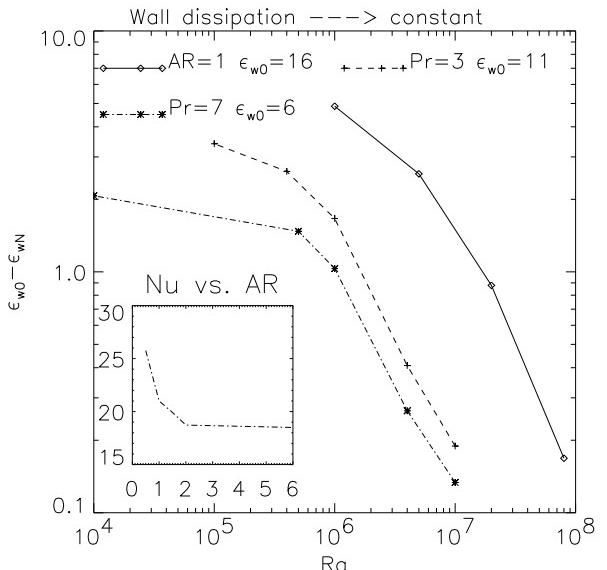


FIG. 2.  $\epsilon_{W0}(AR, Pr) - \epsilon_N$  vs.  $Ra$ . The decrease is approximately  $Ra^{-0.8 \pm 0.1}$  for all three cases (neglecting the lowest  $Ra$  for  $Pr = 0.3$  and 7). The values of  $\epsilon_{W0}(AR, Pr)$  used are given. The inset shows the dependence on aspect ratio of  $Nu$  for  $Ra = 10^7$  and  $Pr = 0.7$ .

While this analysis has confirmed that  $\epsilon_W$  is much larger than the average dissipation and  $\epsilon_W/\epsilon_T$  is increasing with  $Ra$ , careful examination revealed that there appears to be an upper bound to  $\epsilon_W/\epsilon_T$ , which will be denoted  $\epsilon_{W0}$  and is found to be strongly dependent upon  $Pr$  and  $AR$ . Fig. 2 shows the normalized dissipation at the wall plotted as  $\epsilon_{W0} - \epsilon_{WN}$ . For all three cases,  $\epsilon_{W0} - \epsilon_{WN} \sim Ra^{-0.8 \pm 0.1}$ . While the choice of this particular form is subjective, the consistency in  $\epsilon_{WN} \rightarrow \epsilon_{W0}(AR, Pr)$  as  $Ra$  grows for all  $AR$  and  $Pr$

appears to be robust and contradicts the predictions of both the turbulent and laminar boundary layer theories. Due to the bound (2) on  $\epsilon_{BL}$ , then  $\epsilon_{BL}/\epsilon_B \rightarrow 0$  as  $Ra \rightarrow \infty$  at least as fast as

$$\epsilon_{BL}/\epsilon_B < \epsilon_{W0}\lambda_{BL} \sim Ra^{-1/7} \quad (4)$$

where  $-1/7$  comes from using  $\lambda_{BL} = \lambda_u$  from (3).

If this trend were to continue to higher  $Ra$  it would imply that the boundary layer should not be characterized as either a laminar or a turbulent shear-driven boundary layer as has been assumed up to now. Furthermore, it implies that the smallest length scales in the problem are multiples of the Kolmogorov scale  $\eta = \epsilon_T^{-1/4} = (Ra(Nu - 1))^{-1/4}$ . For example, the wall boundary layer thickness taken from the wall shear stress  $z^+ = \sqrt{\tau}$  should scale as  $\eta$ . If  $\beta_T = 0.309$ , then  $\beta^+ = 0.327$  is predicted.

Now let us consider the mechanisms responsible for transferring energy from the bulk to the boundary layer. This is necessary because Fig. 1 shows that the production of kinetic energy, which is equivalent to the convective heat flux  $\overline{w\theta}$ , is found only in the center of the box, but the peak of the dissipation is at the wall. For there to be a turbulent boundary layer, there would have to be large turbulent production or shear production terms. With the normalization used in Fig. 1,  $\overline{w\theta}/(RaNu) \approx 1$  in the center and  $\epsilon(z)/(RaNu)$  is slightly less than 1, which is compensated for by extra dissipation in the boundary layer.

The three transport terms in (1) that transfer energy from the bulk to the boundary layer are diffusive transport  $Pr\partial_z^2 q^2$ , the pressure transport  $-\overline{pw}_{,z}$  and the turbulent transport  $-0.5wq^2_{,z}$ .  $Pr\partial_z^2 q^2$  and  $-0.5wq^2_{,z}$  can be calculated directly while  $-\overline{pw}_{,z}$  can be calculated from the difference between all the remaining terms. Diffusive transport (not shown) is found to be large only very near the wall, for  $1-z < 0.02$  in Fig. 1. For  $1-z < 0.1$  the turbulent transport is much less than the difference between the production and the dissipation, and therefore the turbulent transport is much less than the pressure transport term. This would be consistent with the dissipation in the boundary layer being much less than either the predictions of a turbulent or a laminar boundary layer, both of which depend upon shear production, which is part of the turbulent production term.

The observation that the pressure transport dominates is not new. In the geophysical literature over a wide range in  $Ra$  beginning at low  $Ra$  in the laboratory [3] and extended to higher  $Ra$  in atmospheric observations [10,11], it is found that the primary source of kinetic energy in the boundary layer is the pressure transport term.

If the boundary layer is neither a shear driven turbulent nor a laminar boundary layer, then what mechanism is responsible for the observed scaling of the velocity scale or Reynolds number  $Re = wd/\nu$ ? Perhaps the origin is in the details of plume dynamics. It has previously been

found [16], and confirmed by these simulations, that there is a nearly perfect correlation in the bulk between the vertical velocity  $w$  and temperature fluctuations  $\theta$ , where the total temperature is  $T(x, y, z) = \overline{T}(z) + \theta(x, y, z)$ . What is found everywhere in these calculations, except near the wall ( $z < \lambda_T = d/Nu$ ), is that

$$\overline{\theta w} \approx 0.9\overline{w^2}^{1/2}\overline{\theta^2}^{1/2} \quad (5)$$

This would be consistent with visualizations that show small plumes dominating in the boundary layer [5,14,15].

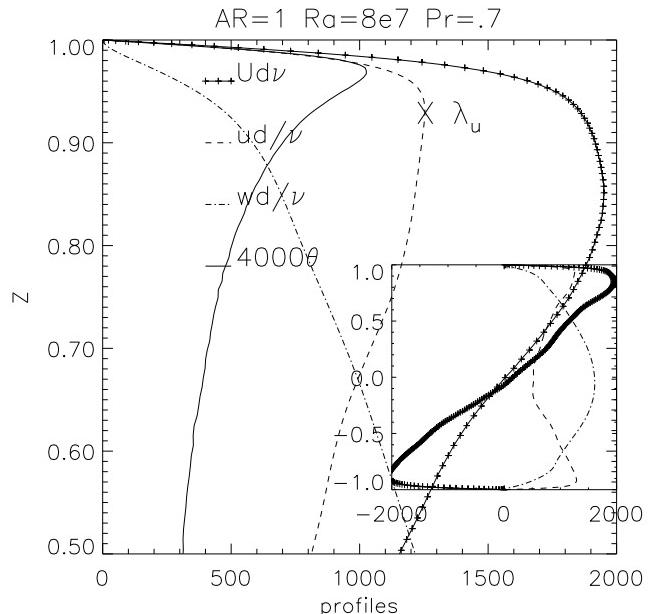


FIG. 3. Reynolds numbers based upon the mean horizontal velocity in the direction of maximum shear  $Vd/\nu$ , upon the horizontal fluctuations in velocity  $ud/\nu = (u^2 + v^2)^{1/2}d/\nu$ , upon the vertical velocity fluctuations  $wd/\nu$ , and the temperature fluctuation variance  $\overline{\theta^2}^{1/2}$ . The inset shows the profiles over the entire domain.

Using (5), it can be predicted that in the center that if  $Nu = \overline{\theta w}/\kappa(d\Theta/dz) \sim Ra^{\beta_T}$ ,  $\theta' = \overline{\theta^2}^{1/2} \sim Ra^{-\delta_c}$ , and  $Re = \overline{w^2}^{1/2}/\nu d \sim Ra^\gamma$  that

$$\gamma = \beta_T + \delta_c. \quad (6)$$

The scaling for the simulations and experiments discussed here is given in Table I. For the experimental values [2] of  $\beta_T = 0.309$  and  $\delta_c = 0.145$ , then (6) would predict  $\gamma = 0.454$ , which is the origin of the value given in Table 1. However, what is found (private communication) for the most recent experiment [2] and most earlier experiments [17] is  $\gamma = 0.5$ . For  $AR = 4$ ,  $Pr = 0.3$  and  $AR = 1$ ,  $Pr = 0.7$ :  $\gamma = 0.46$  and for  $AR = 4$ ,  $Pr = 7$ :  $\gamma > 0.5$ .

These inconsistencies can be resolved by looking at the scaling of the maximum vertical velocity in the simulations, where it is found that  $Re_{max} \sim Ra^{0.5}$  for all three

simulated cases. Furthermore, this maximum is always found roughly midway up a sidewall in a strong persistent plume. Midway up a sidewall is also where the experimental measurements of velocity are taken. Therefore, this indicates that these experimental velocity measurements are not representative of the velocity as a whole, but only representative of the velocity at the walls that probably comes from the maximum possible vertical velocity within plumes. The reason  $AR = 4$ ,  $Pr = 7$  gives  $\gamma > 0.5$  can be understood by noting that even at its highest  $Ra$  of  $10^7$ , the  $Re$  for this case is not turbulent and that in the published visualizations [15] this case is dominated by laminar plumes. Therefore, the velocity in individual plumes seems to obey  $\gamma_{max} = 0.5$ , but when the flow is turbulent the average exponent is closer to  $\gamma = 0.46$ . This could represent an average between  $\gamma_{max} = 0.5$  and the exponent found for the Reynolds number dependence of the large-scale circulation, where  $\gamma = 0.43$  [18].

Expon	Expt. $AR = 1$ , $Pr = 0.7$	$4.0.3$	$4.7$
$\beta_T$	0.309	$0.27 \pm 0.02$	$0.31 \pm 0.02$
$\gamma_{max}$	0.50	$0.50 \pm 0.01$	$0.50 \pm 0.01$
$\delta_c$	0.145	$0.17 \pm 0.02$	$0.14 \pm 0.02$
$\gamma$	0.454	$0.46 \pm 0.02$	$0.46 \pm 0.01$
$Re(Ra = 10^7)$		434	3000
			14

TABLE I. Experimental [2] and numerical exponents. Aspect ratio and Prandtl number for the three numerical cases is given. The experimental value for velocity scaling is given as  $\gamma_{max}$ . How the experimental value for  $\gamma$  is gotten is explained in the text. Errors for the simulations are based upon scatter of exponents taken between  $Ra$ .  $Re = ud/\nu$  is taken at  $Ra = 10^7$  for all three cases.

It has been found that the dissipation at the wall approaches a multiple of the mean dissipation across the box, contrary to the assumptions of the effect of shear upon the thermal boundary layer. Therefore, it does not seem that the dynamics in the boundary layer can be governed by shears, whether they be laminar or turbulent. Instead, it is suggested that plumes might dominate the dynamics and perhaps a new model of turbulent convection should be constructed based upon plume dynamics. The basis for this suggestion is how the velocity scales. Evidence is presented that the experimentally observed scaling of the Reynolds number as  $Re \sim Ra^\gamma$ , with  $\gamma = 0.5$  can be reproduced by the simulations only if this is taken to be the maximum vertical velocity in persistent plumes along the sidewalls. So one should replace  $\gamma$  by  $\gamma_{max}$  in the experiments. Based upon the simulations, it is proposed that the simulated exponent based upon the average kinetic energy is  $\gamma \approx 0.46$ , which is shown to be consistent with the experimental measurements of the scaling of the temperature fluctuations and  $Nu$ . This suggests that it would be useful to have several determinations of the Reynolds numbers from a single experiment. That is, Reynolds numbers based upon velocity measurements taken near a sidewall, in the interior, and from the large-scale circulation.

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